OPTIMAL COMBINATION OF CONTROL AND OBSERVATION

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Optimal control under incomplete information, i.e., under incomplete and inexact measurements of the state of the controlled plant, is of essential interest for practical control problems. In the case when the observer can modify, along with the control, also the measurement program, there arises various problems of the optimal choice of the control and of the observation process, one of which we consider in this paper. We show that under assumptions made on the characteristics of the control systems, Bellman's method and the maximum principle method can be applied for solving the problem under investigation, allowing us to find in a number of cases an explicit form of the optimal control and observation procedure. We remark that for deterministic systems the problem of the optimal combination of control and observation has been examined in monographs [1, 2] in a formulation other than the one adopted here.

1. Let us denote the controlled motion of the plant being investigated by $x(t)$, and the variable accessible to measurement by $y(t)$. Suppose that the vector $x(t)$, taking values in an Euclidean space E_n, is a solution of the system of stochastic differential equations

$$dx(t) = [A(t)x(t) + B(t)u(t)]\,dt + \sigma_1(t)\,d\xi_1(t)$$
$$x(0) = x_0 \quad (0 < t \leqslant T) \tag{1.1}$$

and that the vector $y(t) \in E_m$ is given by the relations

$$dy(t) = H(t)x(t)\,dt + \sigma(t)\,d\xi_2(t) \quad (t > 0), \quad y(0) = 0 \tag{1.2}$$

Here, Eqs. (1.1), (1.2) are to be understood in the sense of Ito [3], and the vectors encountered in them are to be considered as column-vectors. We always assume that the following constraints are fulfilled for the coefficients of Eqs. (1.1), (1.2). The independent Wiener processes $\xi_1(t) \in E_n$ and $\xi_2(t) \in E_m$ are normalized by the conditions

$$\xi_i(0) = 0, \quad M\xi_i(t) = 0, \quad M\xi_1(t)\xi_1'(t) = I_n t, \quad M\xi_2(t)\xi_2'(t) = I_m t$$

where M stands for the mean, the prime is the sign for the transpose, I_n is the n-dimensional unit matrix. The control vector $u(t) \in E_{n_1}$. The deterministic matrices $A(t)$, $B(t)$, $\sigma_1(t)$, $H(t)$, $\sigma(t)$ have bounded piecewise-continuous elements and, moreover, the matrices $A(t)$, $\sigma_1(t)$ have dimension $n \times n$, while the matrices $B(t)$, $H(t)$, $\sigma(t)$ have dimensions $n \times n_1$, $m \times n$, $m \times m$, respectively. Finally, we take it that the matrix $\sigma(t)$ is nonsingular and that the random variable x_0, not dependent on $\xi_1(t)$, $\xi_2(t)$ has a Gaussian distribution with parameters

$$m_0 = Mx_0, \qquad D_0 = M(x_0 - m_0)(x_0 - m_0)'$$

where the matrix D_0 is positive definite.

Since the moving coordinates of the phase vector $x(t)$ are not accessible to direct measurement, the control $u(t)$ at the instant t has to be chosen in the form of a functional depending on a measured realization $y(s)$ over the interval $0 \leqslant s \leqslant t$.

We describe the observation process. During the observation the matrix $H(t)$, giving the composition of the measurements, and the matrix $\sigma(t)$, defining their accuracy, may vary (within the limits of the above-mentioned constraints) in such a way that under these variations the nonnegative-definite matrix

$$V(t) = H'(t)(\sigma(t)\sigma'(t))^{-1} H(t) \qquad (0 \leqslant t \leqslant T) \tag{1.3}$$

describes a set W of nonnegative-definite deterministic matrices with bounded piecewise-right-continuous elements, which is closed and bounded relative to the Euclidean norm. In formula (1.3), and everywhere subsequently, the symbol X^{-1} denotes the matrix inverse to matrix X. Beyond requirement (1.3) we also take it that for any element

$$V(t) \in W \tag{1.4}$$

there holds the equality

$$\int_0^T f(V(s))\, ds = T_0 \quad (T_0 < T), \quad f(V) = \begin{cases} 0, & V = 0 \\ 1, & V \neq 0 \end{cases} \tag{1.5}$$

In other words, on the basis of the definition of the matrix-valued function $f(V)$, condition (1.5) signifies that the total duration of the observation process has been given. The appropriateness of defining the observation process by matrix (1.3), first noted in [4] for uncontrollable motions $x(t)$, is clear from the fact that after the optimal control is chosen the performance criteria being considered become functions only of the variance of the estimate of vector $x(t)$, defined, precisely, by the matrix $V(t)$. Note that for any preassigned nonnegative definite matrix $V(t)$ there exist a matrix $H(t)$ and a nonsingular matrix $\sigma(t)$, satisfying equality (1.3).

Problem 1. To chose a control $u(t)$ in the form of a functional of $y(s)$, $0 \leqslant s \leqslant t$ and a function $V(t)$, satisfying conditions (1.4), (1.5), so as to minimize the performance criterion

$$J(u, V) = M\left[x'(T) L_1 x(T) + \int_0^T (x'(s) L_2(s) x(s) + u'(s) L_3(s) u(s))\, ds\right] \tag{1.6}$$

Here L_1, L_2, L_3 are given deterministic matrices of dimensions $n \times n$, $n \times n$, $n_1 \times n_1$ respectively, with piecewise-continuous elements, where matrix L_2 is nonnegative definite while the matrices L_1 and $L_3(s)$ (for all $0 \leqslant s \leqslant T$) are positive definite. Thus, to solve Problem 1 it is necessary to find a control $u(t)$, optimal in the sense of performance criterion (1.6), in the form of a synthesizing function, and a deterministic measurement program. If $B(t) \equiv 0$ (i.e., system (1.1) is uncontrollable), then Problem 1 reduces to choosing an optimal measurement program, considered earlier in [4].

For any fixed observation method the problem of synthesizing a control $u(t)$, optimal in the sense of performance criterion (1.6), can be solved by means of Bellman's method and the Kalman-Bucy filter (for example, see [5]). Let us cite the results ([5], pp. 96-102) needed subsequently.

For this purpose we denote by $x_1(t)$ and $D(t)$ respectively, the conditional mean and the conditional variance matrix of process $x(t)$ under the condition that a realization of process $y(s)$ has been measured on the interval $0 \leqslant s \leqslant t$. Then [5], the function $x_1(t)$ is a solution of the system of stochastic differential equations

$$dx_1(t) = [A(t)x_1(t) + B(t)u(t)]\,dt + D(t)H'(t)(\sigma(t)\sigma'(t))^{-1} \times \\ \times [dy(t) - H(t)x_1(t)\,dt], \qquad x_1(0) = m_0$$

while the matrix $D(t)$ is determined by the equalities

$$D^{\cdot}(t) = A(t)D(t) + D(t)A'(t) - D(t)V(t)D(t) + \sigma_1(t)\sigma_1'(t) \quad (t>0) \\ D(0) = D_0 \tag{1.7}$$

The optimal control $u_0(t)$ is

$$u_0(t) = -L_2^{-1}(t)B'(t)g(t)x_1(t) \tag{1.8}$$

where the positive-definite matrix $g(t)$ satisfies the equations

$$g^{\cdot}(t) = -A'(t)g(t) - g(t)A(t) + g(t)B(t)L_2^{-1}(t)B'(t)g(t) - L_2(t) \\ g(T) = L_1 \tag{1.9}$$

Finally,

$$J_1(V) = \min_u J(u,V) = J(u_0,V) = m_0'g(0)m_0 + \\ + \operatorname{Tr} L_1D(T) + \int_0^T \operatorname{Tr}[D(s)V(s)D(s)g(s) + L_2(s)D(s)]\,ds \tag{1.10}$$

Here and subsequently the symbol $\operatorname{Tr} L_1$ denotes the trace of matrix L_1. Thus, to solve Problem 1 it remains to minimize $J_1(V)$ over functions $V(t)$, satisfying requirements (1.4), (1.5).

2. Let us transform the right-hand side of formula (1.10). First of all, with due regard to (1.7), we have

$$\int_0^T [A(s)D(s) + D(s)A'(s) - D^{\cdot}(s) + \sigma_1(s)\sigma_1'(s)]\,g(s)\,ds = \\ = \int_0^T D(s)V(s)D(s)g(s)\,ds \tag{2.1}$$

Further, by integrating by parts, we obtain in view of (1.9),

$$-\int_0^T D^{\cdot}(s)g(s)\,ds = -D(T)g(T) + D(0)g(0) + \int_0^T D(s)\dot g(s)\,ds = \\ = -D(T)L_1 + D_0g(0) + \int_0^T D(s)\dot g(s)\,ds \tag{2.2}$$

For any square matrices A_1, A_2 of like dimension,

$$\operatorname{Tr} A_1A_2 = \operatorname{Tr} A_2A_1 \tag{2.3}$$

therefore, by virtue of (1.10), (2.1), (2.2),

$$J_1(V) = m_0'g(0)m_0 + \operatorname{Tr} D_0g(0) + \int_0^T \operatorname{Tr}[D(s)\dot g(s) + \\ + L_2(s)D(s) + (A(s)D(s) + D(s)A'(s) + \sigma_1(s)\sigma_1'(s))g(s)]\,ds$$

Finally, having substituted here in the place of the derivative $g^{\bullet}(t)$ its expression given by the right-hand side of formula (1.9), we are convinced, using (2.3), that

$$J_1(V) = m_0' g(0) m_0 + \mathrm{Tr}\, D_0 g(0) + $$
$$+ \int_0^T \mathrm{Tr}\, [\sigma_1(s)\sigma_1'(s) + D(s) g(s) B(s) L_3^{-1}(s) B'(s)]\, g(s)\, ds$$

But in the last relation for $J_1(V)$ on the basis of (1.9), only the quantity

$$J_2(V) = \int_0^T \mathrm{Tr}\, D(s) g(s) B(s) L_3^{-1}(s) B'(s) g(s)\, ds \qquad (2.4)$$

depends upon the choice of the function $V(t)$

Consequently, to solve Problem 1, which has been reduced to the determination of a matrix $V(t)$, subject to conditions (1.4), (1.5)and minimizing functional $J_2(V)$, we may apply the usual methods of optimal control. In the next section we find the explicit form of the optimal observation method for certain equations of form (1.1), (1.2) with the aid of the maximum principle (see (1.8)).

3. Suppose that the one-dimensional Eqs. (1.1), (1.2) for the scalar variables $x(t)$, $y(t)$ have the form

$$x^{\bullet}(t) = a(t)\, x(t) + b(t)\, u(t) \quad (0 < t \leqslant T) \qquad (3.1)$$

$$dy(t) = h(t)\, x(t)\, dt + \sigma d\xi_2(t) \qquad (3.2)$$

with a constant coefficient $\sigma \neq 0$ and with a function $h(t)$, equal at any instant t either to zero or to a constant $h \neq 0$. The set W consists of the numbers 0 and $h^2\sigma^{-2}$, while performance criterion (1.6) is

$$M\left(l_1 x^2(T) + \int_0^T [x^2(s)\, l_2(s) + u^2(s)\, l_3(s)]\, ds\right)$$

When these requirements are satisfied there holds

Theorem 3.1. If $P(t : b(t) = 0) = 0$ ($P(\delta)$ is the Lebesgue measure on the direct set δ) and if the function $a(t) \leqslant 0$ for all $t \in [0, T]$, then the optimal observation law $V_0(t)$, solving Problem 1 for system (3.1), (3.2), is determined by the equalities

$$V_0(t) = \begin{cases} h^2\sigma^{-2} & (0 \leqslant t < T_0) \\ 0 & (T_0 \leqslant t \leqslant T) \end{cases}$$

Proof. By virtue of (1.7), (3.1), (3.2) the variance $D(t)$ of the estimate satisfies the equation

$$D^{\bullet}(t) = 2a(t)\, D(t) - D^2(t)\, V(t) \qquad (0 < t \leqslant T) \qquad (3.3)$$

Since $D(0) = D_0 > 0$, $D(t) > 0$ for any finite t and, hence, there exists the function $z(t) = D^{-1}(t)$, defined with due regard to (3.3) for $0 < t \leqslant T$ by the equalities

$$z^{\bullet}(t) = -2a(t)\; z(t) + V(t), \qquad z(0) = D_0^{-1}$$

Thus, to prove the theorem it is enough to establish, on the basis of (2.4), that $V_0(t)$ minimizes the functional

$$J_2(V) = \int_0^T \frac{g^2(s)\,b^2(s)}{z(s)\,l_3(s)}\,ds$$

under the supplementary constraints (1.4), (1.5), (3.3). It is not difficult to show, by applying the maximum principle ([6], pp. 75-79), that

$$V_0(t) = \begin{cases} h^2\sigma^{-2}, & \text{if } \psi(t) + c > 0 \\ 0\,, & \text{if } \psi(t) + c \leqslant 0 \end{cases} \tag{3.4}$$

where the constant c is chosen so as to satisfy requirement (1.5) and the adjoint variable $\psi(t)$, is given by the formulas

$$\psi^{\boldsymbol{\cdot}}(t) = 2a(t)\psi(t) - (z^{-1}(t)g(t)b(t))^2\, l_3^{-1}(t) \qquad (0 < t \leqslant T)$$

$$\psi(T) = 0 \tag{3.5}$$

and equals

$$\psi(t) = \int_t^T \frac{(g(s)\,b(s))^2}{z^2(s)\,l_3(s)} \exp\left(\int_s^t 2a(s_1)\,ds_1\right) ds \tag{3.6}$$

Indeed, we set up the Hamiltonian ([6], p. 76)

$$\psi_0 z^{-1}(t) g^2(t) b^2(t) l_3^{-1}(t) + \psi(t)(-2a(t)z(t) + V(t)) + \psi_2(t)V(t)h^{-2}\sigma^2$$

where the constant $\psi_0 \leqslant 0$ by virtue of the transversality condition $\psi(T) = 0$.Further, the function $\psi_2(t)$, satisfying the equation $\psi_2^{\boldsymbol{\cdot}}(t) = 0$ is constant. Hence we should take $\psi_0 < 0$, because when $\psi_0 = 0$ the identity $\psi(t) \equiv 0$ is valid, while the constant $\psi_2(t)$ is nonzero since on the basis of the maximum principle the vector$(\psi_0, \psi(t), \psi_2(t))$is nontrivial. However, for any choice of the constant $\psi_2(t)$ the equality

$$(\psi_0, \psi(t)\ , \psi_2(t)) = (0, 0, \psi_2(t))$$

contradicts requirement (1.5) since by virtue of the maximum principle the optimal observation law is determined by formulas (3.4) with $c = \psi_2(t)h^{-2}\sigma^2$. It still remains to note that the Hamiltonian has been defined to within a constant factor. Therefore, we can set $\psi_0 = -1$.From equality (3.6) we see that $\psi(t) \geqslant 0$. Consequently, on the basis of (3.5) and of the hypotheses of Theorem 3.1, the derivative $\psi^{\boldsymbol{\cdot}}(t) < 0$, i.e., the function $\psi(t)$ decreases monotonically. From this and from (3.4) ensues the validity of the assertion of Theorem 3.1.

Corollary 3.1. Let us assume that in Eqs. (1.1), (1.2) the matrix $B(t)$, $0 \leqslant t \leqslant T$, is nonsingular, the matrix $A(t) = -\gamma I_n$ (the constant $\gamma \geqslant 0$), $\sigma_1(t) = 0$, and the constant coefficient $\sigma(t) = \sigma$. The function $H(t)$ takes two values: it equals either zero or a constant matrix H.The set Wconsists of the null matrix and the matrix $H_1 = H'(\sigma\sigma')^{-1}H$. By $\varphi(t)$ we denote a function equal to unity if an observation is made at the instant t and to zero if an observation is not made. We show that the optimal observation law is

$$V_0(t) = H_1\,\varphi_0(t)$$

where

$$\varphi_0(t) = \begin{cases} 1, & 0 \leqslant t < T_0 \\ 0, & T_0 \leqslant t \leqslant T \end{cases} \tag{3.7}$$

To do this it suffices to prove that among all the functions $V(t) = H_1\,\varphi(t)$ the matrix $V_0(t)$ minimizes functional (2.4) under the supplementary constraints (1.5), (2.1) which, under the assumptions made on the coefficients, are written in the form

$$D^{\cdot}(t) = -2\gamma I_n D(t) - D(t)V(t)D(t), \qquad D(0) = D_0$$

$$\int_0^T \varphi(s)\,ds = T_0, \qquad T_0 < T \tag{3.8}$$

Note that by virtue of the Jacobi identity ([7], p. 420) the determinant $\det D(t)$ of matrix $D(t)$ satisfies the relation

$$\det D(t) = \det D_0 \exp\left\{\int_0^t \mathrm{Tr}\,(-2\gamma I_n - V(s)D(s))\,ds\right\}$$

from which, with due regard to the nonsingularity of D_0 ensues the nonsingularity of matrix $D(t)$. Consequently, the inverse matrix $z(t) = D^{-1}(t)$, exists and is defined, by virtue of (3.8), by the equations

$$z^{\cdot}(t) = 2\gamma I_n z(t) + V(t), \qquad z(0) = z_0 = D_0^{-1} \tag{3.9}$$

In correspondence with [7](p. 283) we can find a nonsingular matrix Q, which simultaneously takes z_0 into the unit matrix and the matrix H_1 into a diagonal matrix H_2. We denote the diagonal elements of matrix H_2 by the symbols h_i, $i = 1,\ldots, n$. Then, paying further heed to (2.3), (3.9), for functional (2.4) we obtain the expression

$$J_2(V) = \int_0^T \mathrm{Tr}\left[z_0 e^{2\gamma I_n t} + \int_0^t e^{2\gamma I_n (t-s)} V(s)\,ds\right]^{-1} \alpha(t)\,dt =$$

$$= \int_0^T \mathrm{Tr}\left[Q^{-1}Qz_0Q'(Q')^{-1} e^{2\gamma I_n t} + \int_0^t e^{2\gamma I_n (t-s)} Q^{-1}QV(s)\,Q'(Q')^{-1}\,ds\right]^{-1} \alpha(t)\,dt =$$

$$= \int_0^T \mathrm{Tr}\left[e^{2\gamma I_n t} + \int_0^t e^{2\gamma I_n (t-s)} H_2 \varphi(s)\,ds\right]^{-0} Q\alpha(t)\,Q'dt \tag{3.10}$$

where the positive-definite matrix

$$\alpha(t) = g(t)\,B(t)\,L_2^{-1}(t)\,B'(t)\,g(t)$$

Hence it follows that for $0 \leqslant t \leqslant T$ all the functions $\beta_i(t)$, being the diagonal elements of a positive-definite matrix $Q\,\alpha(t)\,Q'$, are positive. Furthermore, on the basis of (3.10), using the diagonality of the matrix

$$e^{2\gamma I_n t} + \int_0^t e^{2\gamma I_n (t-s)} H_2 \varphi(s)\,ds$$

we have

$$\min_V J_2(V) = \min_\varphi \int_0^T \sum_{i=1}^n \beta_i(t)\left[e^{2\gamma t} + \int_0^t e^{2\gamma(t-s)} h_i \varphi(s)\,ds\right]^{-1} dt \geqslant$$

$$\geqslant \sum_{i=1}^n \min_\varphi \int_0^T \beta_i(t)\left[e^{2\gamma t} + \int_0^t e^{2\gamma(t-s)} h_i \varphi(s)\,ds\right]^{-1} dt \tag{3.11}$$

where the equality sign holds here only in the case when all the terms in the right-hand side of (3.11) achieve a minimum for one and the same choice of the function $\varphi(t)$. However, any of the terms indicated can be represented in the form

$$\int_0^T \beta_i(s) f_i^{-1}(s)\, ds \qquad (i=1,\ldots,n) \tag{3.12}$$

where

$$f_i'(t) = 2\gamma f_i(t) + h_i\varphi(t) \qquad (0<t\leqslant T)$$
$$f_i(0) = 1$$

Thus, on the basis of Theorem 3.1 and of the inequalities $\beta_i(t) \geqslant 0$, $i = 1,\ldots, n$ noted above, the function $\varphi_0(t)$ minimizes the whole expression (3.12) and consequently also the functional (2.4).

Note 3.1. By a verbatim repetition of the arguments applied in the proof of Corollary 3.1 we convince ourselves that its assertion remains in force if all the requirements of Corollary 3.1 are satisfied, but the matrix $A(t) = -\gamma(t) I_n$ with a nonnegative function $\gamma(t)$.

Corollary 3.2. Let us assume that in Eqs. (1.1), (1.2) the matrix $A(t) = A$ is constant and nonpositive definite, $\sigma_1(t) = 0$, the function $H(t)$ equals either zero or the constant matrix $\gamma_1 H$, the constant coefficient $\sigma(t) = \gamma_2\sigma$ and moreover the matrices H and σ are orthogonal (i.e., $H'H = \sigma\sigma' = I_m$), $\det B(t) \neq 0$ and the numbers $\gamma_i \neq 0$. Further, let the a priori variance $D_0 = \gamma I_n$, $\gamma > 0$. Then, the optimal observation law is $V_0(t) = \gamma^2{}_1\gamma_2^{-2}\varphi_0(t)$, where the function $\varphi_0(t)$ is determined by equality (3.7).

The proof of this corollary is similar to the proof of Corollary 3.1. Namely, by Q we denote an orthogonal matrix taking A to diagonal form with elements in the diagonal of the reduced matrix $a_i \leqslant 0$. Then the matrix $Q \exp(At) Q'$ also is diagonal with diagonal elements equal to $\exp(a_i t)$. Analogous to (3.10), (3.11) we have

$$J_2(V) = \int_0^T \sum_{i=1}^n \beta_i(t) \left[\gamma e^{2a_i t} + \int_0^t e^{2a_i(t-s)} \gamma_1^2\gamma_2^{-2}\varphi(s)\, ds\right]^{-1} dt$$

By arguing further just as in the proof of Corollary 3.1, we convince ourselves of the validity of Corollary 3.2.

4. Let us now study the form of the optimal observation method for systems (3.1), (3.2) without assuming the negativity of $a(t)$ for a performance criterion (1.6) of the form

$$M\left[l_1 x^2(T) + \int_0^T u^2(s) l_3(s)\, ds\right], \quad l_1 > 0, \quad l_3(t) > 0 \tag{4.1}$$

Theorem 4.1. Assume that the function $b(t)$ and the coefficients of Eq. (3.2) satisfy the requirements of Theorem 3.1. the functions $a(t)$, $b(t)$, $l_3(t)$ are differentiable, and $a^\cdot(t) \leqslant 0$, $l_3^\cdot(t) \leqslant 0$, $b(t)\, b^\cdot(t) \geqslant 0$. Then we can find a number $t_1 \leqslant T - T_0$, such that the optimal observation law is $V_0(t) = h^2\sigma^{-2}\varphi(t, t_1)$ where

$$\varphi(t, t_1) = \begin{cases} 0 & (t<t_1) \\ 1 & (t_1 \leqslant t < t_1 + T_0) \\ 0 & (t_1 + T_0 \leqslant t) \end{cases} \tag{4.2}$$

Proof. On the basis of the maximum principle the optimal observation law is determined by formulas (3.4) - (3.6). We now assume the contrary, i.e., we assume that there exist several intervals, nonabutting on each other, on the interval $[0, T]$ where the function $V_0(t)$, determined by equalities (3.4) is nonzero. We denote the i- th of the intervals indicated (the observations intervals) by the symbol $[t_i, s_i)$. It is clear that by virtue of the assumptions made we can find an i, for which $s_i < t_{i+1}$. Let us investigate the behavior of the adjoint variable $\psi(t)$ (see formulas (3.5), (3.6)) on the interval $[s_i, t_{i+1}]$. First of all we note that since $V_0(t) > 0$ for $t \in (t_i, s_i)$ and $V_0(t) = 0$ on the interval $s_i \leqslant t < t_{i+1}$ with due regard to (3.4) the derivative

$$\psi^{\bullet}(s_i) \leqslant 0 \tag{4.3}$$

Let us find an equation satisfied by the quantity $r(t) = \psi^{\bullet}(t)$ on the interval $s_i < t < t_{i+1}$. For this we differentiate with respect to t both sides of relation (3.5). Keeping in mind formulas (3.4), (1.9), (4.1) and the fact that $V_0(t) = 0$ for $s_i \leqslant t < t_{i+1}$, we obtain

$$\begin{aligned} r^{\bullet}(t) - 2a(t)r(t) = r_1(t) = 2a^{\bullet}(t)\psi(t) + l_3^{\bullet}(t)l_3^{-2}(t)g^2(t)b^2(t)z^{-2}(t) - \\ - 2b(t)b^{\bullet}(t)l_3^{-1}(t)g^2(t)z^{-2}(t) - 2b^4(t)g^3(t)l_3^{-2}(t)z^{-2}(t) \end{aligned} \tag{4.4}$$

On the basis of Eq. (1.9), which under the hypotheses of Theorem 4.1 has the form

$$g^{\bullet}(t) = -2a(t)g(t) + g^2(t)b^2(t)l_3^{-1}(t), \qquad t > 0$$
$$g(T) = l_1 > 0$$

the function $g(t)$, equal to

$$g(t) = \left[\frac{1}{l_1}\exp\left(2\int_T^t a(s)\,ds\right) + \int_t^T b^2(s)l_3^{-1}(s)\exp\left(2\int_s^t a(s_1)\,ds_1\right)ds\right]^{-1} \tag{4.5}$$

is positive. Hence from the nonnegativity of the adjoint variable (3.6) and from the requirements of Theorem 4.1 there follows the validity of the inequality

$$r_1(t) < 0, \qquad s_i \leqslant t \leqslant t_{i+1}$$

Therefore, by virtue of relations (4.3), (4.4) for $s_i < t \leqslant t_{i+1}$

$$r(t) = \psi^{\bullet}(s_i)\exp\left(2\int_{s_i}^t a(s)\,ds\right) + \int_{s_i}^t \exp\left(2\int_s^t a(s_1)\,ds_1\right)r_1(s)\,ds < 0$$

In other words, the function $\psi(t)$ decreases monotonically on the interval $[s_i, t_{i+1}]$. However, this latter is impossible since on the basis of (3.4) it contradicts the assumption made above that $V_0(t) > 0$ for $t_{i+j} \leqslant t < s_{i+j}$, $j = 0, 1$ and $V_0(t) = 0$ for $t \in [s_i, t_{i+1})$. The contradiction obtained proves Theorem 4.1.

With the aid of Theorem 4.1 the problem of finding the optimal observation law which minimizes the functional (2.4) corresponding to (4.1), having the form

$$\int_0^T D(s)g^2(s)b^2(s)l_3^{-1}(s)\,ds = J_3 \tag{4.6}$$

can be reduced to the minimization of a scalar function of the variable t_1. For this we should solve Eq. (3.4), having set (see formula (4.2)) $V(t) = h^2\sigma^{-2}\varphi(t, t_1)$, and then we should substitute this solution of Eq. (3.4) and the function (4.5) into integral (4.6) which, after the substitution indicated, will be a function of the one variable t_1. Let us illustrate what we have said by examples.

Example 4.1. Let the coefficients of Eq. (3.2) satisfy the requirements of Theorem 3.1, let the quantities $a(t) = a$, $b(t) = b \neq 0$ be constants, and let the function $l_3(t) \equiv 1$, in functional (4.1), moreover, let

$$b^2 l_1 = 2a > 0 \tag{4.7}$$

We study the optimal observation law under these constraints. From (4.5), (4.7) it follows that $g(t) \equiv l_1$. Therefore, by substituting into (4.6) the solution of Eq. (3.3) with $V(t) = h^2\sigma^{-2}\varphi(t, t_1)$ it is not difficult to get that

$$l_1^{-2}J_3 = \frac{1}{2a} D_0 (e^{2at_1} - 1) - \ln \frac{D_0}{2a} + \ln (e^{2at_1+2aT_0} + 2aD_0^{-1} - e^{2at_1}) + \\ + (e^{2aT} - e^{2a(t_1+T_0)}) [2aD_0^{-1} + e^{2a(t_1+T_0)} - e^{2at_1}]^{-1} \tag{4.8}$$

We set

$$\lambda = e^{2at_1}, \qquad 1 \leqslant \lambda \leqslant e^{2a(T-T_0)}$$

and find that value λ_0, for which the right-hand side of relation (4.8) takes a minimal value. We equate the derivative J_3 with respect to λ to zero. After simple manipulations we have

$$\lambda^2 (e^{2aT_0} - 1) + 2a\lambda D_0^{-1} (1 + e^{2aT_0}) - 2aD_0^{-1}e^{2aT} = 0$$

Hence, by investigating the sign of the derivative $dJ_3 \mid d\lambda$, we convince ourselves that

$$\lambda_0 (e^{2aT_0} - 1) = - aD_0^{-1} (1 + e^{2aT_0}) + [a^2D_0^{-2} (1 + e^{2aT_0})^2 + 2 (e^{2aT_0} - 1) aD_0^{-1}e^{2aT}]^{1/2} \tag{4.9}$$

This formula defines the clear dependency of the starting time of the observations on the system parameters. In particular, from (4.9) we see that for the values $\lambda_0 \leqslant 1$ the observation starting time $t_1 = 0$ and furthermore, $\lim_{a\to\infty} t_1 = T - T_0$.

Example 4.2. Let the coefficients of Eq. (3.2) satisfy the requirements of Theorem 3.1, let the constant quantities $a(t) = a > 0$, $l_3(t) \equiv 1$, $b(t) = b \neq 0$. Let us show that then for all $a > 0$, satisfying the inequalities

$$2a < b^2, \qquad 2ae^{-4aT} \geqslant b^2 (1 - e^{-2aT})^2 \tag{4.10}$$

which are fulfilled for any sufficiently small values of a, the observation starting time $t_1 = 0$. We estimate the derivative $\psi^\bullet (t)$ of the adjoint variable (3.6). In view of (3.6) we have

$$\psi^\bullet (t) = 2ab^2e^{2at} \int_t^T e^{-2as} z^{-2} (s) g^2 (s) ds - g^2 (t) z^{-2} (t) b^2 = \\ = 2ab^2e^{2at} \int_t^T e^{2as} [z (s) e^{2as}]^{-2} g^2 (s) ds - g^2 (t) z^{-2} (t) b^2 \tag{4.11}$$

we now note that on the basis of (3.3) the function $z(t)e^{2at}$, equal to

$$z_0 + \int_0^t e^{2as} V (s) ds$$

does not decrease monotonically for any observation law. From this and from (4.11) ensues the inequality

$$\psi^{\cdot}(t) \leqslant b^2 z^{-2}(t) \left[2ae^{-2at}\int_t^T e^{-2as} g^2(s)\,ds - g^2(t)\right] \tag{4.12}$$

Into the right-hand side of estimate (4.12) we substitute in the place of the function $g(t)$ its expression in (4.5). Integrating with due regard to (4.10), we get

$$\psi^{\cdot}(t) < z^{-2}(t)\, b^2 \left(\frac{b^2}{2a} - 1\right)\left[-e^{-4aT} + \frac{b^2}{2a}(1 - e^{-2aT})^2\right] \leqslant 0$$

In other words the function $\psi(t)$ decreases monotonically, i.e., in view of (3.4) the observation starting time $t_1 = 0$. We remark that, as was shown in [4], for uncontrollable motions the start of the optimal observation process $t_1 = T - T_0$ for any $a > 0$. Thus, the introduction of the control leads to a displacement of the observation interval.

Note 4.1. Similarly to Corollary 3.1 we can formulate simple generalizations of Theorem 4.1 to the multidimensional case. We cite one of them as an example. In relations (1.1), (1.2) let the matrices $A(t) = \gamma_1 I_n$, $D_0 = \gamma_2 I_n$ (γ_1, γ_2 are positive constants), the constant orthogonal matrix $B \equiv B(t)$, let the coefficient $H(t)$, $\sigma(t)$ satisfy the requirements of Corollary 3.1, while in the performance criterion (1.6) let the matrices $L_1 = I_n$, $L_2 \equiv 0$, $L_3 = I_n$. Then there exists only one observation interval. The validity of this fact is established completely analogously to Corollary 3.1 with the use of Theorem 4.1.

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